UNIVERSITÉ DU LUXEMBOURG
ANALYSE 1
2015-2016

EXERCISE SHEET 4

5.1. Let $f : [a, b] \to [a, b]$ be a continuous function. Prove that f has a fixed point.

5.2. Let $f, g : [0, 1] \to \mathbb{R}$ be continuous functions such that

$$f(0) = g(1) = 0$$
$$f(1) = g(0) = 1$$

Prove that for every $\lambda \in \mathbb{R}^+$, there exists $x_\lambda \in [0, 1]$ such that $f(x_\lambda) = \lambda g(x_\lambda)$.

5.3. Let $f : [0, +\infty) \to \mathbb{R}$ be a countinuous function such that $\lim_{x\to+\infty} f(x) = a < \infty$. Prove that f is bounded.

5.4. Let $f, g : [a, b] \to \mathbb{R}$ be continuous functions. Suppose that $f(x) > g(x)$ for every $x \in [a, b]$. Prove that there exists $\lambda > 0$ such that $f(x) \geq g(x) + \lambda$ for every $x \in [a, b]$.

5.5. Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function. Suppose that $\lim_{x\to\infty} \frac{f(x)}{x} = \ell < 1$. Prove that f has a fixed point.

5.6. An athlete walks 6km in an hour. Show that there exists an interval of time of half an hour, in which he walks exactly 3km.

5.7. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous and periodic function. Show that f has a global maximum and minimum.

5.8. Show that the equation $x^2 - 2 = \arctan x$ has at least two solutions.

5.9. Let $\alpha \in [0, 1]$ and consider the following functions:

$$f_\alpha(x) = \begin{cases} x^2 & x \leq 0 \\ \log(x^{2\alpha} + 1) & x > 0 \end{cases}$$

Determine for which values of α, the functions f_α are continuous and differentiable.